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EVERY L_1-PREDUAL IS COMPLEMENTED IN A SIMPLEX SPACE

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ABSTRACT

We show that every L_1-predual space is complemented in a simplex space. This answers a question raised by Lazar and Lindenstrauss.

1. Introduction

This paper is concerned with the question how general L_1-predual spaces are interrelated with simplex spaces. By a simplex S we always mean a compact Choquet simplex; the corresponding simplex space is defined to be

$$A(S) = \{ f: S \to \mathbf{R} : f \text{ affine and continuous} \}.$$

(For simplicity, we assume all Banach spaces to be real.) It is well known that for any separable L_1-predual space X there is a separable simplex space $A(S) \supset X$ and a contractive projection $P: A(S) \to X$ [5]. Lazar and Lindenstrauss posed the question whether this remains true without the assumption of separability.

We give a positive answer to this question. Furthermore, we investigate the geometric relationship between such a simplex and the corresponding dual unit ball $B(X^*)$ of the given L_1-predual space.

There are in fact many similarities between the convexity theory of simplices and L_1-unit balls $B(X^*)$ endowed with the w^*-topology. For example, both admit versions of Michael's selection theorem and versions of Edward's separation theorem (compare [3, Theorem 3.1] with [5, Theorem 2.2] and

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[1, 7.6 Theorem] with [5, Theorem 2.1]). This suggests that $B(X^*)$ is always the 'odd' part of a simplex.

For a Banach space Z let $\Sigma : Z \to Z$ be an isometric, involutive linear map (called *involution*). Put

$$\operatorname{odd} z = \tfrac{1}{2}(z - \Sigma(z)) \quad \text{and} \quad \operatorname{even} z = \tfrac{1}{2}(z + \Sigma(z)) \qquad \text{for any } z \in Z.$$

Then odd and even are contractive projections onto $\operatorname{Odd} Z = \{\operatorname{odd} z : z \in Z\}$ and $\operatorname{Even} Z = \{\operatorname{even} z : z \in Z\}$, resp. Furthermore we have

$$Z = \operatorname{Odd} Z \oplus \operatorname{Even} Z.$$

By $B(Z)$ we mean the closed unit ball of Z, $\operatorname{ex} B(Z)$ are the extreme points of $B(Z)$ and $\partial B(Z)$ are the elements of norm one in Z.

We obtain

THEOREM. *Let X be an L_1-predual space. Then there is a simplex space $Y \supset X$ and an isometric involutive map $\Sigma : Y \to Y$ such that $X = \operatorname{Odd} Y$ and $\Sigma(e) = e$ where e is the one-function in Y.*

COROLLARY I. *Let X be an L_1-predual such that $X^* = l_1$. Then the simplex space Y of the theorem can be taken to satisfy $Y^* = l_1$, too*

COROLLARY II. *Every L_1-predual space X is 1-complemented in a simplex space $A(S)$. $A(S)$ can be taken to have the same density character as X.*

COROLLARY III. *Let B be the unit ball of a conjugate L_1-space (endowed with the w^*-topology). Then there is a simplex S and a surjective continuous affine map $q : S \to B$ satisfying the following*:

(i) $q(\operatorname{ex} S) = \operatorname{ex} B \cup \{0\}$ *and* $q^{-1}(\operatorname{ex} B) \subset \operatorname{ex} S$,

(ii) $q_{|\operatorname{ex} S}$ *is injective*,

(iii) $\operatorname{ex} S \setminus q^{-1}(\operatorname{ex} B)$ *is a singleton*.

In the finite dimensional case one can even replace $\operatorname{ex} B \cup \{0\}$ by $\operatorname{ex} B$ in (i) of the preceding corollary. For example, one can place a tetrahedron S over a rhombus B such that the orthogonal projection $q : \mathbf{R}^3 \to \mathbf{R}^2$ maps S onto B and $\operatorname{ex} S$ bijectively onto $\operatorname{ex} B$.

A slightly more restrictive version of the separable case of the theorem was proven, by different methods, in [8]. We postpone the proofs to Section 3. Here we recall some basic facts concerning simplex spaces [5]. It is well known that an L_1-predual space X is a simplex space if and only if $\operatorname{ex} B(X) \neq \emptyset$. In this case the L_1-order of X^* is the dual order with respect to the pointwise order of the affine functions in X. Moreover, the positive cone of X^*, X_+^*, is then w^*-closed and the positive cap $S = X_+^* \cap B(X^*)$ is a simplex. We have $\mathrm{ex}\, S \subset (\mathrm{ex}\, B(X^*) \cap X_+^*) \cup \{0\}$. (For general L_1-predual spaces X_+^* is not w^*-closed.)

Finally, we recall that an L_1-predual X which is the dual of another Banach space Y is always a $C(K)$-space where $C(K) = \{f: K \to \mathbf{R} : f \text{ continuous}\}$, K a compact Hausdorff space. Moreover, in this case, whenever $T: A \to X$ is a linear bounded operator on some Banach space A and $B \supset A$, there is a norm preserving extension $\tilde{T}: B \to X$ of T [6].

2. L_1-spaces with involutions

In the following we consider an L_1-space V such that there is a subspace V_0 with $V = (V_0 \oplus V_0)_{(1)}$. We put $\Sigma(x, y) = (y, x)$ for all $x, y \in V_0$. Then V_0 is an L_1-space, too. Let the order in V be defined by $(x, y) \geqq 0$ iff $x \geqq 0$ and $y \geqq 0$ (with respect to the L_1-order in V_0). We obtain that

$$\mathrm{Odd}\, V = \{(x, -x) : x \in V_0\}, \qquad \mathrm{Even}\, V = \{(x, x) : x \in V_0\}.$$

The first three of the following lemmas are similar to some results of [2] which were stated and proven in a different context. To make the paper self-contained we include complete proofs.

2.1. Lemma. *If $v, w \in V$ are positive such that* $\| v \| = \| w \| = \| \mathrm{odd}\, v \|$ *and* $\mathrm{odd}\, v = \mathrm{odd}\, w$, *then* $v = w$.

Proof. There are $x, y, z, u \geqq 0$ such that $v = (x, y)$, $w = (z, u)$. By assumption $(x - y, y - x) = (z - u, u - z)$. We obtain $x + u = y + z$. Since all elements involved are positive and V_0 is lattice ordered there are $s_{i,j} \geqq 0$ such that

$$x = s_{1,1} + s_{1,2}, \qquad y = s_{1,1} + s_{2,1},$$

$$u = s_{2,1} + s_{2,2}, \qquad z = s_{1,2} + s_{2,2}.$$

This implies, in view of the fact that the norm on the positive cone is additive,

$$\| v \| = \| x \| + \| y \| = 2 \| s_{1,1} \| + \| s_{1,2} \| + \| s_{2,1} \| .$$

By assumption this is equal to

$$\| \mathrm{odd}\, v \| = \| x - y \| = \| s_{1,2} - s_{2,1} \| .$$

The triangle inequality yields $s_{1,1} = 0$. Similarly we obtain $s_{2,2} = 0$. Hence $x = z$, $y = u$. This proves Lemma 2.1. □

2.2. LEMMA. (a) *If* $w \in \mathrm{Odd}\, V$ *then there is a positive* $v \in V$ *with* $\mathrm{odd}\, v = w$ *and* $\| w \| = \| v \|$.

(b) *V is spanned by the positive elements v such that* $\| v \| = \| \mathrm{odd}\, v \|$.

PROOF. (a) Put $w = (x, -x)$, where $x \in V_0$. Then take $v = 2(x_+, x_-)$. v is positive and we obtain

$$\| v \| = 2 \| x_+ \| + 2 \| x_- \| = 2 \| x \| = \| w \|$$

and $\mathrm{odd}\, v = (x_+ - x_-, x_- - x_+) = w$.

(b) Consider an arbitrary element $(x, y) \in V$. Then

$$(x, y) = (x_+, 0) + (0, y_+) - (x_-, 0) - (0, y_-).$$

Each element v of the form $v = (z, 0)$ or $v = (0, z)$, z positive, satisfies $\| v \| = \| \mathrm{odd}\, v \|$ (since $\mathrm{odd}(z, 0) = \frac{1}{2}(z, -z)$, $\mathrm{odd}(0, z) = \frac{1}{2}(-z, z)$). □

By Lemmas 2.1, 2.2 we can define a map $\rho : \mathrm{Odd}\, V \to V$ as follows. For a given $v \in \mathrm{Odd}\, V$ let $\rho(v)$ be the unique positive element of V such that $\mathrm{odd}\, \rho(v) = v$ and $\| \rho(v) \| = \| v \|$. It is easily seen that $\mathrm{odd} \circ \rho = \mathrm{id}_{\mathrm{Odd}\, V}$ and that ρ is affine on each face of the unit ball of Odd V.

2.3. LEMMA. *For any positive* $v \in V$ *there is a positive* $w \in \mathrm{Even}\, V$ *such that*

$$v = \rho(\mathrm{odd}\, v) + w.$$

PROOF. Let $v = (x, y)$, where $x, y \geqq 0$, and put $z = x - y$. Then $\mathrm{odd}\, v = \frac{1}{2}(z, -z)$. Clearly $x \geqq z$. Hence $x \geqq z_+$. Similarly, $y \geqq z_-$. Put $w = (x - z_+, y - z_-)$. Then w is positive. We have

$$\mathrm{odd}\, w = \tfrac{1}{2}(x - y - (z_+ - z_-), y - x + (z_+ - z_-)) = 0.$$

Hence $w \in \mathrm{Even}\, V$. Finally, we have $\rho(\mathrm{odd}\, v) = (z_+, z_-)$. This implies $v = \rho(\mathrm{odd}\, v) + w$. □

Now we consider V^*. Since $V = (V_0 \oplus V_0)_{(1)}$, we have $V^* = (V_0^* \oplus V_0^*)_{(\infty)}$. Let Σ^* be the adjoint of Σ. Then $\Sigma^*(x^*, y^*) = (y^*, x^*)$ and Σ^* is an isometric involution on V^*.

In view of our remarks at the end of Section 1, V_0^*, V^* are $C(K)$-spaces which have the Hahn–Banach property since V_0, V are L_1-spaces. The order of the $C(K)$-space coincides with the dual order of V. Let e_0 be the one-function in the $C(K)$-space V_0^*. Then $e := (e_0, e_0)$ is the one-function in the $C(K)$-space

V^*. We obtain $\Sigma^*(e) = e$. Note, e corresponds to the functional in V^* which is one on $\{v \in V : \| v \| = 1, v \geqq 0\}$.

For any $\lambda \in \mathbf{R}$ and $v^* = (x^*, -x^*) \in \mathrm{Odd}\ V^*$ we obtain

$$\| \lambda e + v^* \| = \max(\| \lambda e_0 + x^* \| , \| \lambda e_0 - x^* \|) = | \lambda | + \| x^* \| = | \lambda | + \| v^* \| .$$

Here we have

$$(\mathrm{Odd}\ V)^* \cong \mathrm{Odd}(V^*) = \{(x^*, -x^*) : x^* \in V_0^*\}.$$

That is, the restriction map $v^* \mapsto v^*_{|\mathrm{Odd}\ V}$ is an isometry from $\mathrm{Odd}(V^*)$ onto $(\mathrm{Odd}\ V)^*$. So we can identify $(\mathrm{Odd}\ V)^*$ with $\mathrm{Odd}(V^*)$.

In the following lemma we consider an L_1-subspace $U_0 \subset V_0$ and put $U = (U_0 \oplus U_0)_{(1)}$. Hence

$$U \subset V, \quad \Sigma(U) = U, \quad \mathrm{Odd}\ U \subset \mathrm{Odd}\ V.$$

We assume that U_0 is positively embedded in V_0. That is, if $u \in U_0$ is positive with respect to the L_1-order in U_0 it is positive with respect to the L_1-order in V_0.

As before we define $(u_1, u_2) \geqq 0$ in U iff $u_1 \geqq 0$, $u_2 \geqq 0$ in U_0. An operator $T : V \to U$ is called *positive* if $Tv \geqq 0$ whenever $v \geqq 0$ with respect to the corresponding orders.

2.4. Lemma. *Let $P : \mathrm{Odd}\ V \to \mathrm{Odd}\ U$ be a contractive projection. Then there is a positive contractive projection $\hat{P} : V \to U$ such that*

$$\hat{P}_{|\mathrm{Odd}\ V} = P \quad \textit{and} \quad \hat{P} \circ \Sigma = \Sigma \circ \hat{P}.$$

Proof. Consider the adjoint map $P^* : \mathrm{Odd}\ U^* \to \mathrm{Odd}\ V^*$. Let e_U, e_V be the one-functions of U^* and V^*, resp. From the remarks preceding Lemma 2.4 we infer that

$$\mathrm{span}(\{e_U\} \cup \mathrm{Odd}\ U^*) = (\mathbf{R}\{e_U\} \oplus \mathrm{Odd}\ U^*)_{(1)},$$

$$\mathrm{span}(\{e_V\} \cup \mathrm{Odd}\ V^*) = (\mathbf{R}\{e_V\} \oplus \mathrm{Odd}\ V^*)_{(1)}.$$

Put $Q_1(\lambda e_U + u^*) = \lambda e_V + P^* u^*$, where $u^* \in \mathrm{Odd}\ U^*$. Then Q_1 is contractive and $\Sigma^* \circ Q_1 = Q_1 \circ \Sigma^*$. V^* has the Hahn–Banach property, that is, Q_1 has a contractive extension $Q_2 : U^* \to V^*$. We can assume without loss of generality that $Q_2 \circ \Sigma^* = \Sigma^* \circ Q_2$. (Otherwise take $\frac{1}{2}(Q_2 + \Sigma^* \circ Q_2 \circ \Sigma^*)$ instead of Q_2.)

Then $\| Q_2 \| = 1$ and $Q_2 e_U = e_V$. This implies that Q_2 is positive with respect to the dual orders.

The adjoint Q_2^* maps V^{**} into U^{**}. Regard V and U as subspaces of V^{**} and U^{**}, resp., in the usual way. This means

$$U_0 \subset U_0^{**}, \quad V_0 \subset V_0^{**} \qquad \text{and}$$

$$U^{**} = (U_0^{**} \oplus U_0^{**})_{(1)}, \quad V^{**} = (V_0^{**} \oplus V_0^{**})_{(1)}.$$

We have

$$\Sigma^{**} Q_2^* = Q_2^* \Sigma^{**}.$$

All Banach spaces involved are L_1-spaces. It is well known that there is a positive contractive projection $R_0 : U_0^{**} \to U_0$. (This follows e.g. from [7; Proposition 1.a.2 and Lemma 1.b.9].)

Let $R : U^{**} \to U$ be defined by

$$R(x^{**}, y^{**}) = (R_0 x^{**}, R_0 y^{**}).$$

Then R is a positive contractive projection from U^{**} onto U and we have $\Sigma R = R \Sigma^{**}$. Put $\hat{P} = (RQ_2^*)_{|V}$. Then $\hat{P}$ is positive, contractive and extends P. Furthermore $\Sigma \hat{P} = \hat{P} \Sigma$.

It remains to show that $\hat{P}$ is a projection onto U. To this end consider $u \in U$ such that $u \geqq 0$ and $\| u \| = \| \text{odd } u \|$. Here u is positive with respect to U. Since, by assumption, U is positively embedded in V, u is also positive with respect to V. We have

$$\hat{P}(\text{odd } u) = P(\text{odd } u) = \text{odd } u.$$

Since $\text{odd}(\hat{P}u) = \hat{P}(\text{odd } u)$ and $\hat{P}$ is positive we obtain, by Lemma 2.1, that $\hat{P}u = u$. By Lemma 2.2, $\hat{P}_{|U} = \text{id}_U$.

On the other hand, by definition, $\hat{P}V \subset U$. This proves that $\hat{P}$ is a projection onto U. □

3. Proof of the Theorem and the Corollaries

3.1. PROOF OF THE THEOREM. Put

$$V_0 = X^{***}, \quad V = (V_0 \oplus V_0)_{(1)} \quad \text{and} \quad U_0 = X^*, \quad U = (U_0 \oplus U_0)_{(1)};$$

X^{***} and X^* are L_1-spaces. Here X^* is regarded as a subspace of X^{***} in the usual way. Hence U is a subspace of V. Since X^{***} and X^* are L_1-spaces, U_0 is positively embedded in V_0.

Let $P_0 : X^{***} \to X^*$ be the canonical projection, that is

$$(P_0 x^{***})(x) = x^{***}(x) \qquad \text{for all } x \in X \quad \text{and} \quad x^{***} \in X^{***}.$$

Put $P(x^{***}, y^{***}) = (P_0 x^{***}, P_0 y^{***})$.

Apply Lemma 2.4 to obtain a contractive positive extension $\hat{P}: V \to U$ of $P_{|\text{Odd } V}$ such that $\hat{P}$ is a projection onto U. Note that $P_{|\text{Odd } V}$ is the canonical projection since Odd $V = \tilde{X}^{***}$ and Odd $U = \tilde{X}^*$ where

$$\tilde{X} = \{(x, -x) \in (X \oplus X)_{(\infty)} : x \in X\};$$

$\tilde{X}$ is isometrically isomorphic to X and $\tilde{X}^*$ is embedded in $\tilde{X}^{***}$ in the usual way.

But $\hat{P}$ is not necessarily the canonical projection from $(X \oplus X)^{***}_{(\infty)}$ onto $(X \oplus X)^*_{(\infty)}$. (The canonical projection would not be positive in general.)

Let V_+ and U_+ be the positive cones of V and U, resp. Then $U_+ \subset V_+$ since U is a sublattice of V [7; Proposition 1.a.2]. Hence $\hat{P}(V_+) = U_+$. V_+ is closed with respect to $\sigma(V, (X^{**} \oplus X^{**})_{(\infty)})$ since $(X^{**} \oplus X^{**})_{(\infty)}$ is a $C(K)$-space, hence a simplex space. But U_+ is not necessarily closed with respect to $\sigma(U, (X \oplus X)_{(\infty)})$.

We define another w^*-topology on U under which U_+ is closed. Consider on V the topology $\sigma(V, (X^{**} \oplus X^{**})_{(\infty)})$ and let τ be the finest locally convex topology on U such that $\hat{P}$ is continuous. Then the absolutely convex subsets $O \subset U$ such that $\hat{P}^{-1}(O)$ are zero neighbourhoods with respect to $\sigma(V, (X^{**} \oplus X^{**})_{(\infty)})$ form a zero neighbourhood base with respect to τ. τ is Hausdorff since $\hat{P}$ is a projection onto U.

Since $\hat{P} \circ \Sigma = \Sigma \circ \hat{P}$ and $\hat{P} \circ \Sigma$ is continuous, we obtain that $\Sigma: U \to U$ is continuous with respect to τ. Hence odd : $U \to U$ is continuous with respect to τ. Define

$$S = U_+ \cap B(U).$$

Since $\hat{P}$ is continuous, $V_+ \cap B(V)$ is $\sigma(V, (X^{**} \oplus X^{**}))_{(\infty)}$-compact and $\hat{P}(V_+ \cap B(V)) = S$, we obtain that S is τ-compact. Hence S is a Choquet simplex as cap of the positive cone of an L_1-space.

We claim that τ and $\sigma(\text{Odd } U, \tilde{X})$ coincide on $B(\text{Odd } U)$. In this case we find an isometric embedding $T: X \to A(S)$, namely

$$(Tx)(u) = (\text{odd } u)(x, -x),$$

if $x \in X$, $u \in S \subset U = (X^* \oplus X^*)_{(1)}$. As a consequence of Lemma 2.2(a) we obtain that

$$\text{Odd}(S) = B(\text{Odd } U).$$

This shows T is an isometry.

Furthermore we can define

$$(\bar{\Sigma} f)(s) = f(\Sigma s)$$

where $f \in A(S)$, $s \in S$. Then $\bar{\Sigma}$ is an isometric involution on $A(S)$ with $\bar{\Sigma}(1_s) = 1_s$.

If $f \in A(S)$ is such that $f(0) = 0$ then f has a unique τ-continuous extension $\hat{f}$ to an affine function on $B(U)$ with $\hat{f}(-u) = -\hat{f}(u)$. (In fact, if

$$u = \lambda s_1 - (1-\lambda)s_2, \qquad s_1, s_2 \in S, \quad 0 \leqq \lambda \leqq 1,$$

then $\hat{f}(u) = \lambda f(s_1) - (1-\lambda) f(s_2)$.) Provided the claim is true, odd $\hat{f}$ (with respect to $\bar{\Sigma}$) is $\sigma(\mathrm{Odd}\ U, \tilde{X})$-continuous on $B(\mathrm{Odd}\ U)$. Hence it corresponds to an element in X. If $x^* \in B(X^*)$ let $s(x^*) \in S$ be the unique element with

$$\mathrm{odd}\, s(x^*) = \tfrac{1}{2}(x^*, -x^*) \quad \text{and} \quad \| s(x^*) \| = \| x^* \| .$$

For $f \in A(S)$ put

$$x^*(Rf) = \tfrac{1}{2} f(s(x^*)) - \tfrac{1}{2} f(\Sigma(s(x^*))) = (\mathrm{odd}\, \hat{f})(\mathrm{odd}\, s(x^*)).$$

Then $R : A(S) \to X$ is contractive and $RT = \mathrm{id}_X$. Hence $Q := TR$ is a contractive projection onto TX and $TX = \mathrm{Odd}\, A(S)$ (with respect to $\bar{\Sigma}$).

It remains to prove the claim. To this end, recall odd is τ-continuous and we have $\Sigma(B(U)) = B(U)$. Moreover

$$\mathrm{odd} \circ \hat{P} = P \circ \mathrm{odd}, \qquad \begin{array}{ccc} B(V) & \xrightarrow{\hat{P}} & B(U) \\ \downarrow \scriptstyle\mathrm{odd} & & \downarrow \scriptstyle\mathrm{odd} \\ B(\mathrm{Odd}\ V) & \xrightarrow{P} & B(\mathrm{Odd}\ U) \end{array}$$

This shows that odd is continuous with respect to τ on U and $\sigma(\mathrm{Odd}\ U, \tilde{X})$ on Odd U.

Since $B(U) = \hat{P}(B(V))$ is τ-compact we obtain that $B(\mathrm{Odd}\ U)$ is τ-compact. Because odd and $\hat{P}$ restricted to Odd U are the identity, we conclude that the restriction of τ to $B(\mathrm{Odd}\ U)$ is finer than $\sigma(\mathrm{Odd}\ U, \tilde{X})$. Hence both topologies coincide on $B(\mathrm{Odd}\ U)$. This proves the claim and concludes the proof of the theorem. □

3.2. Proof of Corollary III. We retain the notation of 3.1. For $s \in S$ and $x \in X$ put

$$(qs)(x) = (\mathrm{odd}\, s)(x, -x).$$

Then $q : S \to B(X^*)$ is affine, continuous (with respect to $\sigma(X^*, X)$ on $B(X^*)$). q is onto since $\mathrm{Odd}\, S = B(\mathrm{Odd}\, U) \cong B(X^*)$. If $x^* \in \mathrm{ex}\, B(X^*)$ then

$$F := \{s \in S : q(s) = x^*\}$$

is a closed face of S. This implies $\mathrm{ex}\, F \subset (\mathrm{ex}\, S) \setminus \{0\}$. All $s \in F$ satisfy

$$\mathrm{odd}\, s = \tfrac{1}{2}(x^*, -x^*) \quad \text{and} \quad 1 = \| s \| = \| x^* \| .$$

By Lemma 2.1, s is unique. This proves $q^{-1}(\mathrm{ex}\, B(X^*)) \subset \mathrm{ex}\, S$ and shows that q is injective on $q^{-1}(\mathrm{ex}\, B(X^*))$.

Finally we have $q(0) = 0$. If $s \in (\mathrm{ex}\, S) \setminus \{0\}$ then $s \in B(U)$. Hence, by Lemma 2.3, $s = \rho(\mathrm{odd}\, s) + w$ for some positive $w \in \mathrm{Even}\, U$. We obtain

$$1 = \| s \| = \| \rho(\mathrm{odd}\, s) \| + \| w \|$$

since all elements are positive elements of an L_1-space. This shows that $w \in S = U_+ \cap B(U)$. Since s is an extreme point of $B(U)$ we obtain $s = w$ or $s = \rho(\mathrm{odd}\, s)$. If s were even we would have $s = \tfrac{1}{2}(x^*, x^*)$ for some $x^* \in \partial B(X^*)$. Since $s \geqq 0$ we would obtain

$$x^* \geqq 0 \quad \text{and} \quad s = \tfrac{1}{2}(x^*, 0) + \tfrac{1}{2}(0, x^*).$$

This would contradict $s \in \mathrm{ex}\, B(X)$. Hence $s = \rho(\mathrm{odd}\, s)$.

By definition, $\mathrm{odd}\, s = \tfrac{1}{2}(q(s), -q(s))$. If, for some $0 < \lambda < 1$ and $y^*, z^* \in B(X^*)$, $q(s) = \lambda y^* + (1 - \lambda) z^*$, then

$$s' := \lambda \rho(\tfrac{1}{2}(y^*, -y^*)) + (1 - \lambda) \rho(\tfrac{1}{2}(z^*, -z^*))$$

is such that $\mathrm{odd}\, s' = \mathrm{odd}\, s$. Lemma 2.1 again shows $s' = s$. We conclude

$$\rho(\tfrac{1}{2}(y^*, -y^*)) = s \quad \text{or} \quad \rho(\tfrac{1}{2}(z^*, -z^*)) = s.$$

This implies

$$q(s) = y^* = q(\rho(\tfrac{1}{2}(y^*, -y^*))) \quad \text{or} \quad q(s) = z^* = q(\rho(\tfrac{1}{2}(z^*, -z^*))).$$

Hence $q(s) \in \mathrm{ex}\, B(X^*)$. This shows

$$q(\mathrm{ex}\, S) = \mathrm{ex}\, B(X^*) \cup \{0\}.$$

Thus q is injective on $\mathrm{ex}\, S$ and

$$\mathrm{ex}\, S = q^{-1}(\mathrm{ex}\, B(X^*)) \cup \{0\}. \qquad \square$$

3.3. Proof of Corollary I. If $X^* \cong l_1$ then $\mathrm{ex}\, B(X^*)$ is countable. By Corollary III, $\mathrm{ex}\, S$ is countable. Hence $A(S)^* \cong l_1$. $\square$

3.4. PROOF OF COROLLARY II. We use the notation of the theorem. Let $Q = \text{odd}$. Then $Q : Y \to X$ is a contractive projection. Put $W = \text{span}(\{e\} \cup X)$. Then, by [6], there is an L_1-predual $W \subset Z \subset Y$ with the same density character as X. Since $e \in Z$ we obtain $\text{ex}\, B(Z) \neq \emptyset$. Hence Z is a simplex space and $Q_{|Z}$ is a contractive projection onto X. □

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REFERENCES

1. L. Asimov and A. J. Ellis, *Convexity Theory and its Applications in Functional Analysis*, Academic Press, 1980.
2. E. G. Effros, *On a class of real Banach spaces*, Isr. J. Math. **9** (1971), 430–458.
3. A. J. Lazar, *Spaces of affine continuous funtions on simplexes*, Trans. Am. Math. Soc. **134** (1968), 503–525.
4. A. J. Lazar, *On Lindenstrauss spaces*, Proceedings of the Conference on Split Faces, Facial Structures of Compact Convex Sets and Applications, NATO Advanced Studies Institute, University College of Swansea, 1972, pp. 68–72.
5. A. J. Lazar and J. Lindenstrauss, *Banach spaces whose duals are L_1-spaces and their representing matrices*, Acta Math. **126** (1971), 165–194.
6. J. Lindenstrauss, *Extensions of compact operators*, Memoirs Am. Math. Soc. No. **48** (1964).
7. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II*, Springer, 1979.
8. W. Lusky, *On simplex spaces with involutions*, Proceedings of the Conference on Functional Analysis/Banach Space Geometry, Vorlesungen aus dem Fachbereich Mathematik der Universität Essen, Heft 10, 1983, pp. 309–317.